Theory and Measurement of Q in Resonant Ring Circuits*

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Summary—**The loaded and unloaded Q of a resonant ring circuit are derived on the basis of the fundamental definition. Simple experiments are described to measure Q_0, Q_L, and the ring power gain without additional coupling to the ring. A number of graphs are given which are useful for these measurements.**

Introduction

THE traveling-wave resonator or resonant ring circuit has been described by various authors [1–6]. Both directional and nondirectional coupling have been treated, as well as the effect of a lossless discontinuity in the resonant ring. Some applications for the resonant ring circuit have been given which show the general interest in the device [1], [2], [7], [8]. One important characteristic of the circuit is that the power level in the resonant ring may be higher than in the primary transmission line.

A few of the papers go briefly into a discussion of the Q of the circuit [3], [5]. These calculations are of an approximate nature, valid under the assumption of very low loss in the resonant ring. It is the purpose of this paper to derive more accurate relations for the unloaded and loaded Q of the circuit. The paper will further describe simple measurements for Q_0, Q_L, and the ring power gain without additional coupling to the ring. This may be important in cases where the ring circuit wave cannot be readily monitored for reasons of size, or operation in a vacuum where a second coupling element should be avoided. The power loss in the ring circuit can also be determined by this measurement, since the power gain depends on the total losses in the ring. This information may be valuable, if the resonant ring is used to test certain components under high power levels and where the power loss in the component depends on the power level.

Only the case of directional coupling will be treated in this paper. The ideas relating to the determination of Q, the ring power gain, and the ring power loss can easily be applied to nondirectional coupling.

Basic Equations

Two basic configurations of the ring circuit are possible as originally described by Miloševic and Vautey [2]. They are shown schematically in Figs. 1 and 2 and will be referred to as types I and II, respectively. The coupling element between the primary line and the

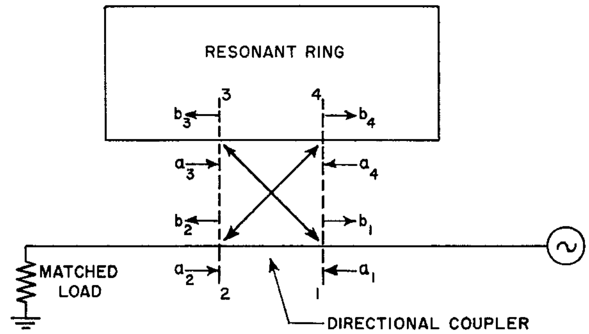

Fig. 1—Resonant ring circuit, type I.

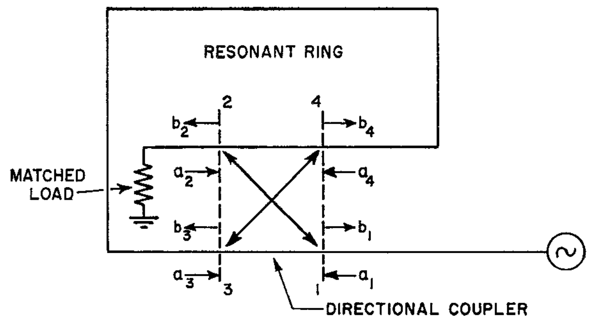

Fig. 2—Resonant ring circuit, type II.

ring circuit is a symmetrical dual directional coupler; other coupling elements like a magic tee or a circulator are equally possible.

No reflections are assumed to exist anywhere, the external load is matched to the transmission line, and the directional coupler has infinite directivity. To simplify the calculations, it will be further assumed that the coupling region between the reference planes is lossless and of infinitesimal length; any loss or phase shift resulting from the finite length of the directional coupler will be absorbed into the connecting transmission lines.

The relations between the incident and reflected waves at the reference planes of the coupler are uniquely determined by the matrix equation

$$b = S \cdot a, \qquad (1)$$

where a is the matrix of the incident waves $a_1 \cdots a_4$, b the matrix of the reflected waves $b_1 \cdots b_4$, and S the scattering matrix of the directional coupler. For a symmetrical, lossless directional coupler, the scattering matrix is given by

* Received by the PGMTT, May 20, 1960; revised manuscript received June 29, 1960.
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$$S = \begin{vmatrix} 0 & k_1 & jk_2 & 0 \\ k_1 & 0 & 0 & jk_2 \\ jk_2 & 0 & 0 & k_1 \\ 0 & jk_2 & k_1 & 0 \end{vmatrix}, \tag{2}$$

where k_1 and k_2 are real due to the assumed infinitesimal length of the coupling region [9].

Conservation of energy demands that

$$k_1^2 + k_2^2 = 1. \tag{3}$$

It is common practice to express the amount of coupling between the primary and secondary branch of a directional coupler in db. From (1) and (2) and Figs. 1 and 2, it follows that

$$\text{Coupling} = 20 \log k_2 \quad \text{db (Type I circuit)} \tag{4a}$$

$$\text{Coupling} = 20 \log k_1 \quad \text{db (Type II circuit)}. \tag{4b}$$

The two resonant ring circuits are sufficiently similar to be treated by the same equations. From the previous assumption, that no reflections occur anywhere, it follows for both circuits that

$$a_2 = a_3 = b_1 = b_4 = 0. \tag{5}$$

The relation between the wave amplitudes a_4 and b_3 is given by

$$a_4 = b_3 e^{-\gamma l}, \tag{6}$$

where

$$\gamma = \alpha + j\beta \tag{7}$$

is the propagation constant in the ring circuit of length l between the reference planes 3 and 4. It is convenient to define a loss factor A by

$$A = e^{-\alpha l}. \tag{8}$$

Eqs. (1), (2), and (6) can be solved for the power gain G between reference planes 3 and 1.

$$G = \left|\frac{b_3}{a_1}\right|^2 = \frac{1 - k_1^2}{|1 - k_1 A e^{-j\beta l}|^2}. \tag{9}$$

The power gain is a maximum if the phase shift around the ring is equal to $2\pi n$, where n is an integer; this resonant power gain is given by

$$G_r = \frac{1 - k_1^2}{(1 - k_1 A)^2}. \tag{10}$$

It is possible to optimize the power gain for a given value of A by a suitable choice of k_1.

$$G_{r,\text{opt}} = \frac{1}{1 - A^2} \tag{11}$$

with

$$k_{1,\text{opt}} = A. \tag{12}$$

Eq. (10) has been plotted by various authors [1–3]. For graphs of the power gain as a function of other parameters, which will be discussed in the following sections, see Figs. 7 and 8.

The Unloaded Q

In general, the unloaded Q of a resonant cavity is defined by

$$Q_0 = \frac{\omega\,(\text{energy stored in cavity})}{\text{power loss in cavity}} = \frac{\omega W}{P_L}, \tag{13}$$

evaluated at resonance. The power balance for the ring circuit at resonance is given by

$$P_L = |a_1|^2 - |b_2|^2, \tag{14}$$

where $|a_1|^2$ is the power delivered by the generator and $|b_2|^2$ is the power dissipated in the external load. The power flow around the ring can be expressed in terms of the power level at reference plane 3 as

$$P(x) = |b_3|^2 e^{-2\alpha x}. \tag{15}$$

The total stored energy in the loop is given by

$$W = \int_3^4 \frac{|b_3|^2 e^{-2\alpha x}}{v_g}\, dx = \frac{|b_3|^2}{2\alpha v_g}(1 - A^2), \tag{16}$$

where v_g is the group velocity in the ring. Then

$$Q_0 = \frac{\omega}{2\alpha v_g} \frac{|b_3|^2 (1 - A^2)}{|a_1|^2 - |b_2|^2}. \tag{17}$$

Introducing relations between the wave amplitudes, (17) can be simplified to

$$Q_0 = \frac{\omega}{2\alpha v_g} = \frac{\pi n \lambda_g^2}{\alpha l \lambda_0^2}, \tag{18}$$

where n is the number of wavelengths in the ring, λ_0 and λ_g are the free space wavelength and the guided wavelength in the ring, respectively. Tischer derived this expression as an approximation for small values of α [3], [5]. However, (18) has been derived from the basic definition of Q_0 without any approximations; it is therefore valid for any value of α.

The Loaded Q

The loaded Q of a resonant circuit is defined by

$$Q_L = \frac{\omega\,(\text{energy stored in cavity})}{\text{power loss in cavity and external circuit}} = \frac{\omega W}{P_L + P_{\text{ext}}}, \tag{19}$$

evaluated at resonance.

Before we evaluate this expression for a resonant ring circuit, it is convenient to consider first the usual single-input resonant circuit, as shown schematically in Fig. 3. Here, the cavity is coupled to the input transmission line by a coupling network. It can easily be shown that the ratio Q_L/Q_0 can be obtained by measuring the voltage standing wave ratio (VSWR) on the line at resonance [10].

$$Q_L = \frac{Q_0}{1+r}, \tag{20}$$

where r is the VSWR for an overcoupled cavity and the reciprocal of the VSWR for an undercoupled cavity. If we introduce the reflection coefficient ρ,

$$|\rho| = \frac{\text{VSWR} - 1}{\text{VSWR} + 1}, \tag{21}$$

(20) reduces to

$$Q_L = \frac{Q_0}{2}(1 \pm |\rho|), \tag{22}$$

where the positive sign is for an undercoupled cavity, the negative sign for an overcoupled cavity. For critical coupling, $\rho=0$, and $Q_L=Q_0/2$.

Referring again to Figs. 1 and 2, it is obvious that the VSWR on the input transmission line is always unity under the idealizing assumptions made in this derivation. The power not coupled into a conventional resonant cavity is reflected toward the generator; the power not coupled into a resonant ring is absorbed in the external load. In order to apply the ideas above to the resonant ring, we must replace the

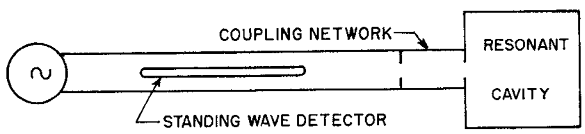

Fig. 3—Schematic diagram of conventional single-input resonant cavity.

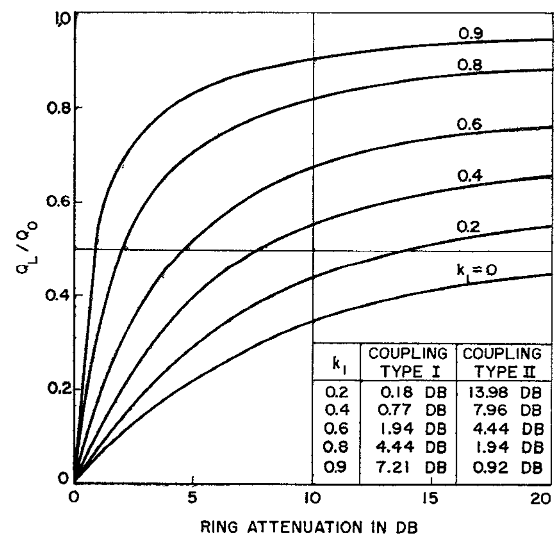

Fig. 4—Q_L/Q_0 vs ring attenuation with the coupling coefficient k_1 as parameter.

reflection coefficient ρ by the wave ratio b_2/a_1, the ratio of the uncoupled wave to the incident wave.

$$Q_L = \frac{Q_0}{2}\left(1 \pm \left|\frac{b_2}{a_1}\right|\right), \tag{23}$$

where again the positive sign refers to an undercoupled cavity, the negative sign to an overcoupled cavity. From the definition of k_1 and k_2 it follows that overcoupling is characterized by $k_1<A$, undercoupling by $k_1>A$. In either case, if the wave ratio b_2/a_1 is evaluated at resonance from (2) and (3) and substituted into (23), the loaded Q is given by

$$Q_L = \frac{Q_0(1+k_1)(1-A)}{2(1-k_1A)}. \tag{24}$$

If the coupling between the primary line and the resonant ring is reduced to zero, *i.e.*, $k_1=1$, $Q_L=Q_0$. For optimum or critical coupling, $k_1=A$, and $Q_L=Q_0/2$. Eq. (24) has been plotted in Fig. 4 as a function of the ring attenuation in db.

An approximate expression for the loaded Q can be derived from (9) for the power gain, since for reasonably large values of Q (19) reduces to

$$Q_L = \frac{\omega}{2\Delta\omega}, \tag{25}$$

where ω is the resonant frequency, and $\omega \pm \Delta\omega$ are the half-power frequencies. Thus,

$$|1 - k_1Ae^{-j\Delta\beta l}|^2 = 2(1-k_1A)^2, \tag{26}$$

assuming that the loss in the ring is constant for small frequency changes. Using the relation

$$\Delta\beta \simeq \frac{\partial\beta}{\partial\omega}\Delta\omega = \frac{\Delta\omega}{v_g}, \tag{27}$$

which is a good approximation for small values of $\Delta\omega$, (26) reduces to

$$\cos\frac{\Delta\omega l}{v_g} = 1 - \frac{(1-k_1A)^2}{2k_1A}. \tag{28}$$

Since (25) is valid only for reasonably large Q_L, the cosine can be approximated by the first two terms of the power series expansion. Then (25) can be written

$$Q_L = Q_0\frac{\sqrt{k_1A}\,\log_e(1/A)}{1-k_1A} \tag{29}$$

Eq. (29) is an approximation for large values of Q_L. It can be compared with (24), which is an exact expression for Q_L. The two expressions agree within a few per cent if the loss in the ring circuit does not exceed 6 db, and k_1 is larger than $\frac{1}{2}$. The latter condition implies that the directional coupler should couple less than 3 db of power from the primary to the secondary branch, if a type I circuit is used, and more than 3 db if a circuit of type II is used. Since directional couplers with more than 3 db coupling are scarcely available, it follows that (29) is not applicable for practical type II circuits. This is not too serious a limitation, since for low losses (less than 6 db) a circuit of type I gives a higher power gain than a circuit of type II with the same directional coupler.

Measurement Procedures

Eq. (23) suggests a simple measurement of the ratio Q_L/Q_0. One possible arrangement is shown in Fig. 5. The two power levels $|b_2|_\omega^2$ and $|a_1|_\omega^2$ are measured independently at the resonant frequency ω by means of two bolometer bridges. If the two auxiliary directional couplers are equal, the two bolometer bridges can be replaced by a reflectometer, which measures the ratio $|b_2/a_1|_\omega$ directly, or by a slotted line. The latter method is shown schematically in Fig. 6. The ratio $|b_2/a_1|_\omega$ can be calculated from the measured VSWR using (21), if $|\rho|$ is replaced by $|b_2/a_1|_\omega$. The resonance in the system is easily determined by noting that the ratio b_2/a_1 is a minimum at resonance.

An independent measurement of the loaded Q is possible if the frequency dependence of the wave b_2 is used. From (1) and (2),

$$\frac{b_2}{a_1} = \frac{k_1 - Ae^{-j\beta l}}{1 - k_1 A e^{-j\beta l}}. \tag{30}$$

At the half-power frequencies, $\omega \pm \Delta\omega$, this wave ratio is given by

$$\left|\frac{b_2}{a_1}\right|_{\omega\pm\Delta\omega} = \left[\frac{k_1^2 + A^2 - 2k_1 A \cos\dfrac{\Delta\omega l}{v_g}}{1 + k_1^2 A^2 - 2k_1 A \cos\dfrac{\Delta\omega l}{v_g}}\right]^{1/2}. \tag{31}$$

Introducing (27), the ratio b_2/a_1 at the half-power points can be expressed by the same ratio measured at resonance.

$$\left|\frac{b_2}{a_1}\right|^2_{\omega\pm\Delta\omega} = \frac{1}{2}\left(\left|\frac{b_2}{a_1}\right|^2_\omega + 1\right). \tag{32}$$

Thus, a measurement of the frequency deviation $\Delta\omega$ which satisfies (32) yields a second independent measurement of Q_L. It should be noted that (32) is valid under the same restrictions as (29).

These two measurements determine both the loaded and the unloaded Q. It is then possible to eliminate both k_1 and A from the expressions for the power gain and express the power gain in terms of the two Q values.

$$G_r = \frac{4\dfrac{Q_L}{Q_0}\left(1 - \dfrac{Q_L}{Q_0}\right)}{1 - \exp\left(\dfrac{-2\pi n v_p}{Q_0 v_g}\right)}. \tag{33}$$

The exponential term generally cannot be neglected. Eq. (33) has been plotted in Fig. 7. This method for measuring the ring power gain is accurate only for reasonably large values of Q_L. Since it is an indirect method, it should be used only in cases where a direct measurement of the ring power gain is not desirable or possible.

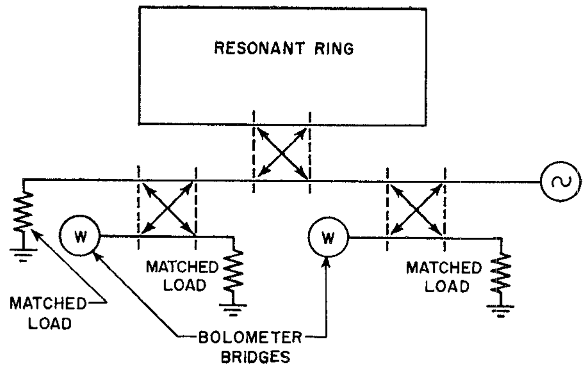

Fig. 5—Schematic diagram for the measurement of Q_L (type I resonant ring).

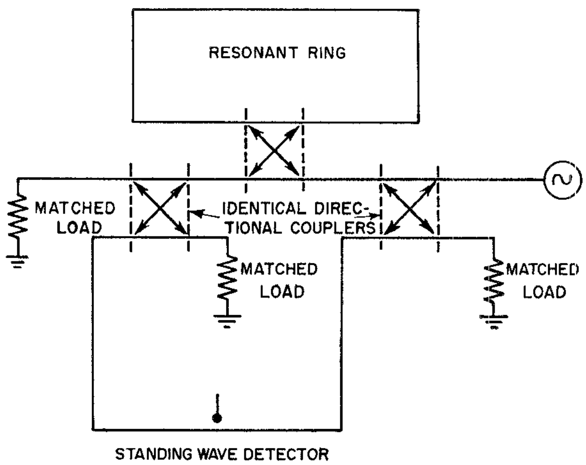

Fig. 6—Alternate setup for the measurement of Q_L (type I resonant ring).

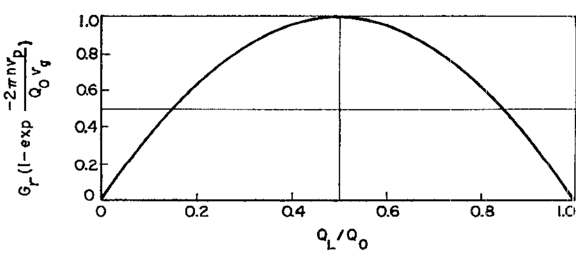

Fig. 7—Normalized ring power gain vs Q_L/Q_0.

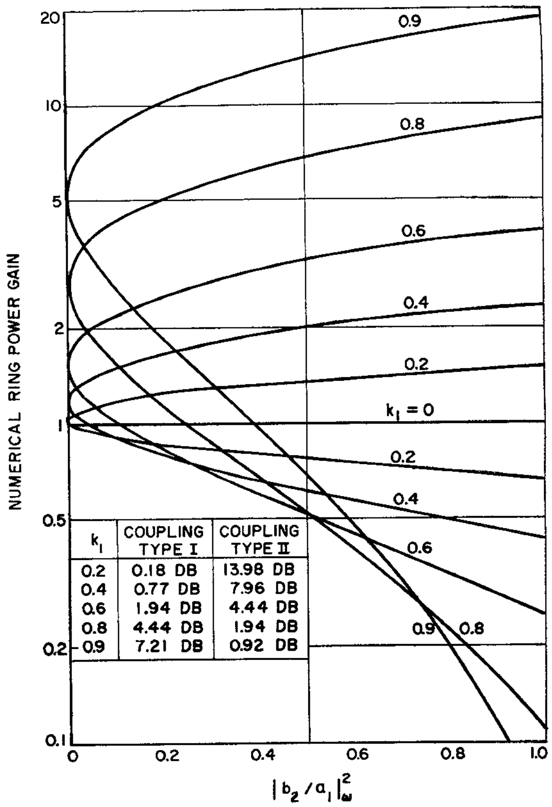

Fig. 8—Resonant ring power gain vs power ratio $|b_2/a_1|_\omega^2$. (Upper branches are for an undercoupled circuit, lower branches for an overcoupled circuit.)

A variation of this method assumes a knowledge of the coupling coefficient k_1. It is then sufficient to measure $|b_2/a_1|_\omega$, the wave ratio at resonance. Eliminating the loss factor A in (10), the power gain can be written in terms of this wave ratio as

$$G_r = \frac{(1 - k_1 | b_2/a_1 |_\omega)^2}{1 - k_1^2} \tag{34a}$$

for an overcoupled circuit, or as

$$G_r = \frac{(1 + k_1 | b_2/a_1 |_\omega)^2}{1 - k_1^2} \tag{34b}$$

for an undercoupled circuit.

The power gain is shown graphically in Fig. 8 as a function of the power ratio $|b_2/a_1|_\omega^2$ with k_1 as a parameter. The upper branch of each curve is for an undercoupled circuit, the lower branch for an overcoupled circuit. This measurement is accurate for all values of the coupling coefficient k_1, since the two equations do not contain any approximations. Since the coupling coefficient can usually be determined accurately, while

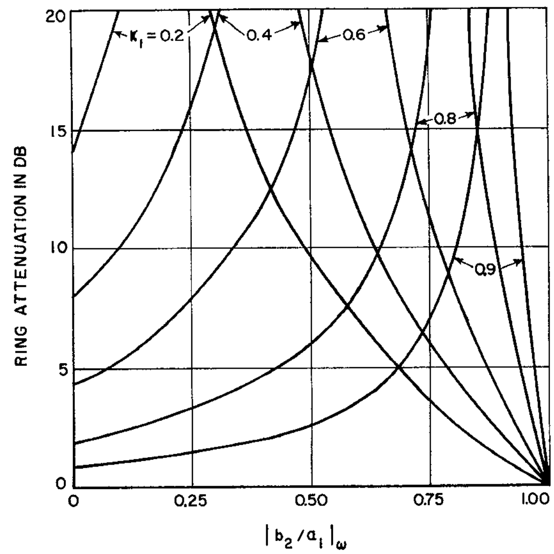

Fig. 9—Ring attenuation vs wave ratio $|b_2/a_1|_\omega$ with k_1 as parameter.

the ring attenuation cannot, this measurement is very often sufficient to determine the power gain.

Note that these methods can be used to determine the ring attenuation. It can easily be shown that

$$A = e^{-\alpha l} = e^{-(\pi n v_p / Q_0 v_g)} = \frac{| k_1 - | b_2/a_1 |_\omega |}{1 - k_1 | b_2/a_1 |_\omega}. \tag{35}$$

This expression is shown graphically in Fig. 9.

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